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A resonance mechanism in plane Couette flow

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The temporal evolution of small three-dimensional disturbances on viscous flows between parallel walls is studied. The initial-value problem is formally solved by using Fourier–Laplace transform techniques. The streamwise velocity component is obtained as the solution of a forced problem. As a consequence of the three-dimensionality, a resonant response is possible, leading to algebraic growth for small times. It occurs when the eigenvalues of the Orr–Sommerfeld equation coincide with the eigenvalues of the homogeneous operator for the streamwise velocity component. The resonance has been investigated numerically for plane Couette flow. The phase speed of the resonant waves equals the average mean velocity. The wavenumber combination that leads to the largest amplitude corresponds to structures highly elongated in the streamwise direction. The maximum amplitude, and the time to reach this maximum, scale with the Reynolds number. The aspect ratio of the most rapidly growing wave increases with the Reynolds number, with its spanwise wavelength approaching a constant value of about 3 channel heights.

1. Introduction

The mechanisms that cause transition to turbulence in parallel shear flows are still not completely understood. A first step towards an understanding of this important problem has been the study of the development of small perturbations on a steady mean flow. The behaviour of such disturbances is generally analysed using properties of the most unstable eigenmode of the Orr–Sommerfeld equation. For the Blasius boundary layer and plane Poiseuille flow, calculations of spatial growth rates and the location of the neutral stability curve are in good agreement with the experimental results obtained with vibrating ribbon techniques (Ross *et al.* 1970; Nishioka, Iida & Ichikawa 1975). Also, the initial stage of the temporal evolution of a three-dimensional disturbance in a laminar boundary layer is well accounted for by the least damped Orr–Sommerfeld mode (cf. Gaster 1975).

If only two-dimensional disturbances are studied, the two velocity components can be obtained from a stream function that is determined from the Orr–Sommerfeld equation. In this case, this equation thus provides a complete description of the perturbation flow field. When three-dimensional disturbances are considered, however, the relation between the velocity components becomes more complex. Then the

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streamwise perturbation velocity, u, is obtained from a forced problem where the vertical velocity, v, and the perturbation pressure act as forcing terms. Since the pressure is linearly related to the vertical velocity, the forcing of the u component is given by the solutions of the Orr–Sommerfeld equation. The response can be determined from the properties of the homogeneous operator of the u component. For bounded shear flows, it turns out that discrete eigensolutions exist. The eigenmodes correspond physically to motions without a v component and without horizontal pressure gradients. They can also be interpreted as vertical vorticity waves. As will be shown later, these waves are always exponentially damped. Were it not for the possibility of resonant driving, these modes would therefore be of little physical significance. The resonance occurs when an eigenvalue of the Orr–Sommerfeld equation coincides with an eigenvalue of the u mode. This leads to a linear growth of the u velocity for small times. Since the resonance can occur only for damped waves, the u component will eventually tend to zero. If the exponential damping rate is small, the maximum amplitude obtained will be large and will occur at a large time.

The present paper is devoted primarily to a numerical study of the proposed resonance mechanism in plane Couette flow. The resonance problem is analytically formulated in terms of an initial value problem for a general bounded, parallel viscous shear flow. The corresponding initial value problem for boundary-layer flows is briefly discussed.

2. Analysis

The non-dimensional equations governing the evolution of small perturbations on a steady parallel flow are

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + U'v = -\frac{\partial p}{\partial x} + \frac{1}{R}\nabla^2 u, \tag{1a}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{R}\nabla^2 v, \tag{1b}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w, \tag{1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

where $(u, v, w), p$ and $[U(y), 0, 0]$ are the perturbation velocities, perturbation pressure and the mean velocity, respectively. $R = U_0\delta/\nu$ is the Reynolds number, where U_0 is a characteristic velocity, δ is the channel height and ν is the kinematic viscosity. The prime denotes differentiation with respect to y and ∇^2 is the Laplacian.

The usual technique for analysing the problem defined above starts with the derivation of a single equation for the vertical velocity component. From (1) and (2), one obtains the following relation between the pressure and the vertical velocity:

$$\nabla^2 p = -2U'\partial v/\partial x. \tag{3}$$

Elimination of the pressure between (1 *b*) and (3) leads to the following equation for the v component:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} = \frac{1}{R}\nabla^4 v. \tag{4}$$

The boundary conditions for v are

$$v = \partial v/\partial y = 0 \quad \text{at} \quad y = 0, 1. \tag{5}$$

The classical Orr–Sommerfeld equation (cf. Lin 1955) can be obtained from (4) by using the normal-mode assumption.

Once the v component is obtained, the pressure can be calculated from (3). Alternatively, the pressure can be determined from the more convenient expression

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\frac{\partial v}{\partial y} - U'\frac{\partial v}{\partial x}, \tag{6}$$

which has been obtained from (1a), (1b) and (2). Finally, the u component can be obtained as the solution to the forced problem (1a), and w can then be calculated from continuity.

An illustrative picture of the driving mechanism is obtained if the forced problem is analysed in terms of the vertical vorticity component. The vorticity equation can be derived by applying the curl operator on (1). In particular, the y component of the vorticity, ω_y, is governed by

$$\frac{\partial \omega_y}{\partial t} + U\frac{\partial \omega_y}{\partial x} - \frac{1}{R}\nabla^2 \omega_y = -U'\frac{\partial v}{\partial z}. \tag{7}$$

Since ω_y is not related to v, the term on the right-hand side acts as a source term for vertical vorticity. The effect of the source term has the following simple physical interpretation.

The perturbation vertical velocity displaces fluid particles. These have streamwise velocities determined by the value of the mean flow at their height of origin. If the displacement has a spanwise variation, adjoining fluid particles will have different streamwise velocities and thus vertical vorticity will be created. Note that this is a truly three-dimensional phenomenon. For two-dimensional disturbances the y component of the vorticity equation is identically zero.

For bounded shear flows, the homogeneous operator in (7) subject to

$$\omega_y = 0 \quad \text{at} \quad y = 0, 1 \tag{8}$$

have wave-like solutions. A resonant response is then possible if a Tollmien–Schlichting wave has the same spatial and temporal behaviour as the free vertical vorticity wave. In that case, the vertical vorticity will grow linearly in time for small times.

In constructing the formal solution to the forced problem, it is preferable to proceed in terms of the velocity components rather than the vertical vorticity since the former are directly measurable quantities. Formal expressions for u, v, w and p can be constructed by using Fourier–Laplace transform techniques. The Fourier transform with respect to the homogeneous co-ordinates x and z, and the Laplace transform with respect to time are defined as

$$\hat{f}(\alpha, y, \beta, t) = \iint_{-\infty}^{+\infty} e^{-i(\alpha x + \beta z)} f(x, y, z, t)\, dx\, dz, \tag{9}$$

$$\tilde{f}(x, y, z, s) = \int_0^{+\infty} e^{-st} f(x, y, z, t)\, dt. \tag{10}$$

Transformation of (1*a*), (4) and (6) yields

$$D^2\psi-[k^2+R(s+i\alpha U)]\psi = R(i\alpha\Pi+U'\phi-\hat{u}_0), \tag{11}$$

$$(D^2-k^2)^2\phi-R[(s+i\alpha U)(D^2-k^2)\phi-i\alpha U''\phi] = -R\widehat{\nabla^2 v_0}, \tag{12}$$

$$\Pi = \frac{1}{k^2R}\{[D^2-k^2-R(s+i\alpha U)]D\phi+R(D\hat{v}_0+i\alpha U'\phi)\}, \tag{13}$$

where ψ, ϕ and Π are the Fourier–Laplace transforms of u, v and p, respectively; $\hat{u}_0$ and $\hat{v}_0$ are the Fourier transforms of the initial conditions. D represents differentiation with respect to y and $k^2=\alpha^2+\beta^2$. The boundary conditions are

$$\psi=\phi=D\phi=0 \quad \text{at} \quad y=0,1. \tag{14}$$

Assume that the solution for ϕ, and thereby also Π, is known. Equation (11) can then be solved by the standard method of variation of parameters. After partial integrations, the expression for ψ becomes

$$\psi=T_1+T_2+T_3, \tag{15}$$

where

$$T_1=\frac{i\beta R}{k^2}\left(\frac{\chi_1}{E}\int_0^1\hat{\omega}_{y0}\chi_2\,dy-\psi_1\int_0^y\hat{\omega}_{y0}\psi_2\,dy_1+\psi_2\int_0^y\hat{\omega}_{y0}\psi_1\,dy_1\right), \tag{16}$$

$$T_2=\frac{\beta^2R}{k^2}\left(\frac{\chi_1}{E}\int_0^1 U'\phi\chi_2\,dy-\psi_1\int_0^y U'\phi\psi_2\,dy_1+\psi_2\int_0^y U'\phi\psi_1\,dy_1\right), \tag{17}$$

$$T_3=\frac{i\alpha}{k^2}\frac{\partial\phi}{\partial y}, \tag{18}$$

$$\chi_1=\psi_1\psi_{20}-\psi_2\psi_{10}, \quad \chi_2=\psi_1\psi_{21}-\psi_2\psi_{11}, \qquad (19),(20)$$

$$E=\psi_{10}\psi_{21}-\psi_{11}\psi_{20}. \tag{21}$$

Here ψ_1 and ψ_2 are the homogeneous solutions to (11) normalized such that their Wronskian equals unity; $\hat{\omega}_{y0}$ is the Fourier transform of the initial vorticity in the y direction. The second subscript in (19)–(21) indicates that the quantities are evaluated at either boundary. The Fourier–Laplace transformed vertical vorticity component can be expressed in terms of (16)–(18) as

$$\tilde{\hat{\omega}}_y=\frac{ik^2}{\beta}(T_1+T_2). \tag{22}$$

For two-dimensional disturbances, the streamwise velocity is obtained directly from continuity. This is also evident in (15)–(18) since then T_1 and T_2 equal zero. When inverting the Laplace transform in the three-dimensional case, two distinct contributions to the streamwise velocity can be identified. These are associated with the poles of ϕ and the poles determined by the roots of

$$E(s;\alpha,\beta,R)=0; \tag{23}$$

ϕ has simple poles in the complex s plane determined by the Orr–Sommerfeld eigenvalue problem. The corresponding eigenmodes can be excited only by vertical motions. The roots to (23) are the eigenvalues of the homogeneous operator in (11) subject to (14). Physically, these eigensolutions are pressureless motions in horizontal

x, z planes. Vertical vorticity disturbances exclusively excite these modes. For two-dimensional motion, these eigenmodes do not have any physical interpretation. In this case, β is equal to zero and the roots to (23) do not lead to poles in ψ.

A resonance phenomenon occurs if a root to (23) coincides with an Orr–Sommerfeld-type pole. The resulting double pole appears in T_2 and leads to a temporal behaviour of the form $te^{s_0 t}$, where s_0 is the pole. Even if s_0 corresponds to an exponentially damped wave, the disturbance grows linearly for small times. The maximum amplitude can become large if the wave is weakly damped. The study of this resonance is the main concern of the rest of this paper.

3. The eigenvalue problems

The resonance phenomenon for three-dimensional disturbances can be investigated by studying the following two eigenvalue problems:

$$D^2\psi - [k^2 + i\alpha R(U - c_1)]\psi = 0, \tag{24}$$

$$\psi = 0 \quad \text{at} \quad y = 0, 1; \tag{25}$$

and

$$(D^2 - k^2)^2\phi - i\alpha R[(U - c_2)(D^2 - k^2)\phi - U''\phi] = 0, \tag{26}$$

$$\phi = D\phi = 0 \quad \text{at} \quad y = 0, 1. \tag{27}$$

Equation (24) is the homogeneous operator appearing in (11) and (26) is the classical Orr–Sommerfeld equation. Resonant driving of the streamwise velocity, or, equivalently, the vertical vorticity, occurs when the eigenvalues c_1 and c_2 coincide for a given wavenumber vector (α, β) and Reynolds number.

Equation (24) is mathematically equivalent to the equation describing the temperature modes for thermally stratified plane Couette flow (Davey & Reid 1977). Following these investigators, multiplication of (24) with the complex conjugate of ψ followed by integration over the channel height leads to the following bounds for the real and imaginary parts of c_1:

$$U_{\min} < c_{1r} < U_{\max}, \tag{28}$$

$$c_{1i} < -\frac{1}{\alpha R}(k^2 + \pi^2). \tag{29}$$

This shows that the vertical vorticity modes always are damped. Similar bounds for c_2 can be found in Joseph (1969).

Equation (24) can be simplified by the substitution

$$c_1 = c' - ik^2/\alpha R \tag{30}$$

which transforms (24) into

$$D^2\psi - i\alpha R(U - c')\psi = 0. \tag{31}$$

The eigenvalue c' depends only on αR. Once this dependency is known, c_1 can be determined for arbitrary combinations of α, β and R through (30).

The eigenvalues to (24) and (26) must, in general, be obtained by using numerical techniques. Numerical results for plane Couette flow will be presented in the following section.

4. Numerical results for plane Couette flow

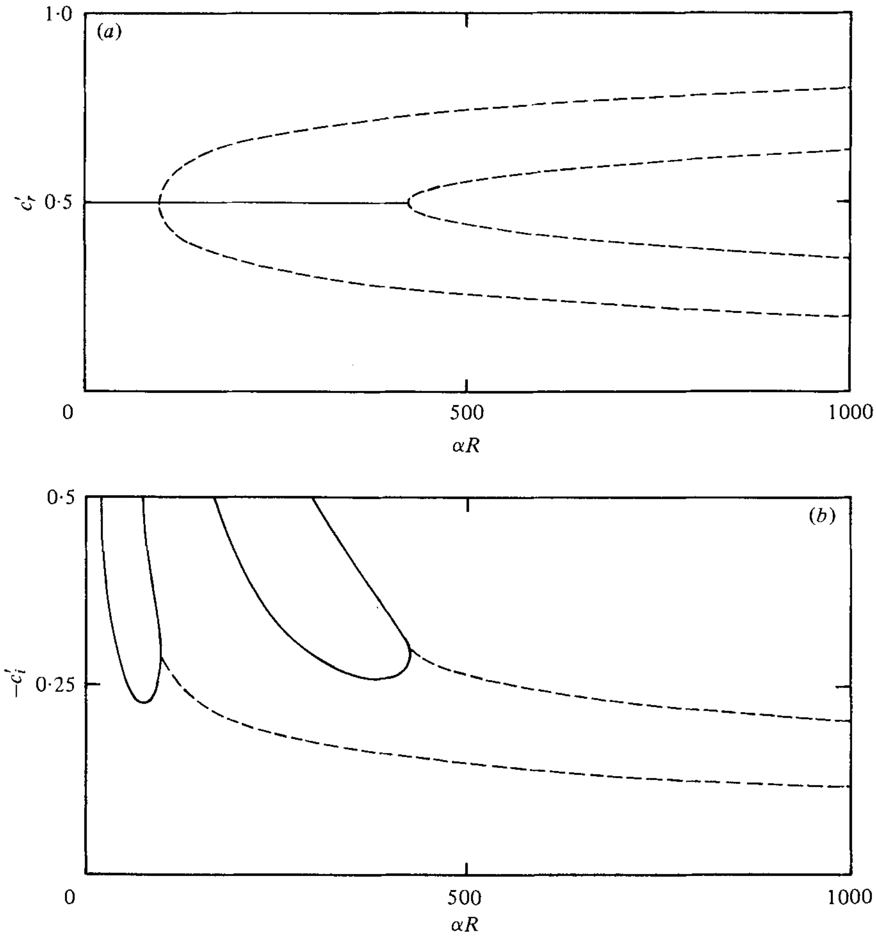


FIGURE 1. The relation between (a) the real part and (b) the imaginary part of the eigenvalue c' and αR for the first four modes. ——, $c_r' = \frac{1}{2}$; – – –, $c_r' \neq \frac{1}{2}$.

The eigenvalues to (24) and (26) were obtained by numerical iteration. The Adams interpolation method (cf. Collatz 1960, p. 126) was used for the numerical integration of both equations. The eigenvalues were determined to six decimal places.

The depencency of c', defined in (31), on αR is shown in figure 1. The results agree with those obtained for the temperature mode by Davey & Reid (1977). In figure 2, the eigenvalues c_2 for the four least damped Orr–Sommerfeld modes are shown as a function of αR with k equal to 1. For k equal to 5, the corresponding results are shown in figure 3. The results agree with those presented by Gallagher (1974). As αR becomes smaller than a certain value, the phase velocities for both types of waves become $\frac{1}{2}$. The resonance is most likely to occur for αR smaller than this value. The numerical study was therefore then concentrated on waves with phase velocities equal to $\frac{1}{2}$. However, a resonance for waves with other wave speeds cannot be completely ruled out. For the case when c_r equals $\frac{1}{2}$, c_i is shown as a function of αR, with k equal to 5, in figure 4. It is seen that the two types of modes have coinciding eigenvalues at two

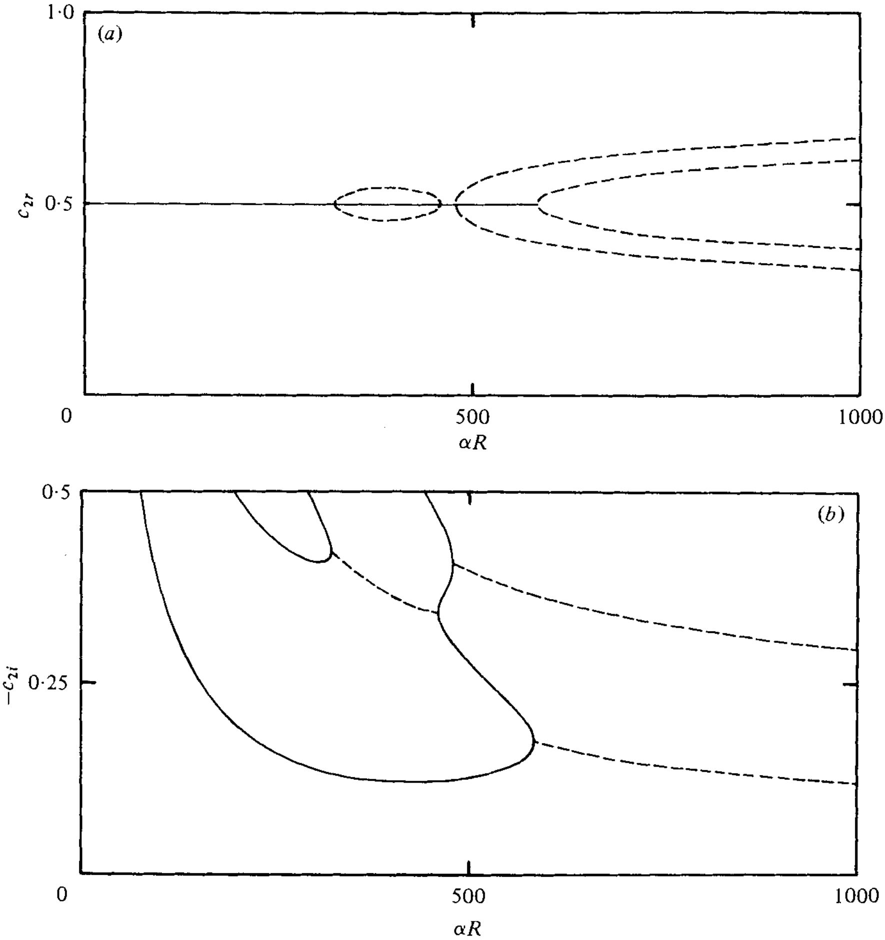
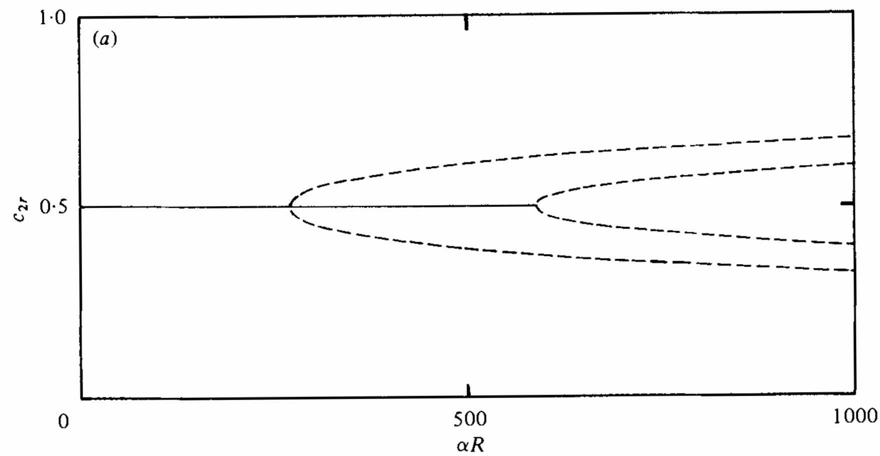


FIGURE 2. The relation between (*a*) the real part and (*b*) the imaginary part of the eigenvalue c_2 and αR for the first four modes, $k = 1$. ——, $c_{2r} = \frac{1}{2}$; ---, $c_{2r} \neq \frac{1}{2}$.

points (P_1 and P_2). Similar curves can be obtained for other values of k. The point P_1 was found always to correspond to the waves with the smallest exponential damping rate. These damping rates, normalized with R^{-1}, are shown as a function of k in figure 5. A perturbation analysis shows that $\alpha c_i R$ approaches $-4\pi^2$ as k tends to zero. It was found that a minimum in $|\alpha c_i R|$ occurs for k approximately equal to 2. For a given Reynolds number, this minimum corresponds to the wavenumber combination for which the maximum amplitude of the resonant wave becomes the largest. The flatness of the minimum indicates that there is a wide range of wavenumber combinations with similar growth properties. In table 1, the characteristic properties of the resonant waves at P_1 are given as a function of k. For R equal to 500, it is noted that the resonant wavenumbers correspond to structures highly elongated in the streamwise direction. The eigenfunction χ_1, defined in (19), at the mode crossing point corresponding to maximum growth is shown in figure 6.

The effect on the results of a change in the Reynolds number can be assessed by noting that both c_1 and c_2 depend only on αR and k. For a given k, the resonance will occur at successively smaller α as R is increased. Since c_i remains a constant, $|\alpha c_i|$ at



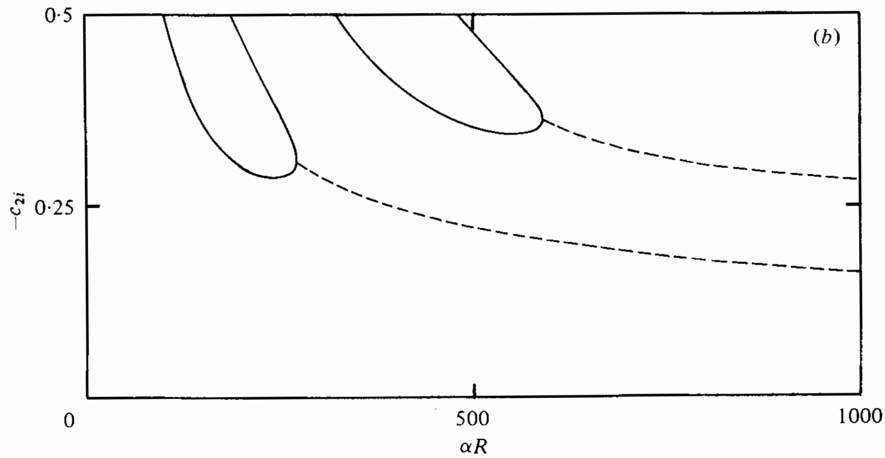


FIGURE 3. The relation between (*a*) the real part and (*b*) the imaginary part of the eigenvalue c_2 and αR for the first four modes, $k = 5$. ——, $c_{2r} = \frac{1}{2}$; – – –, $c_{2r} \neq \frac{1}{2}$.

resonance decreases with increasing Reynolds number. In the limit $R \to \infty$, the resonant growth of the disturbance is linear for all times. This result has recently been obtained by Landahl (1979, private communication) from considerations based on the streamwise averaged inviscid equations. Furthermore, the aspect ratio of the disturbance tends to infinity in this limit. However, the spanwise wavenumber exhibiting the fastest growth tends to approximately 2, which corresponds to a spanwise wavelength of about 3 channel heights.

5. Discussion

The calculations of Gallagher & Mercer (1962, 1964), Davey (1973) and Gallagher (1974) very strongly indicate that the Tollmien–Schlichting waves in plane Couette flow are always damped. Therefore, the resonance mechanism presented here seems to be the only linear process which can produce growth of disturbances in this type of flow. In general, large amplitudes are obtained for structures highly elongated in the streamwise direction. Largely because of difficulties in establishing the mean flow,

k	αR	$\alpha c_i R$	β/α (R = 500)
0	0	−39.480	—
1	66·530	−38·475	7·4486
2	90·500	−37·835	10·9921
3	97·835	−39·165	15·2993
5	95·640	−49·560	26·1206
7	89·080	−70·250	39·2778

TABLE 1. Characteristic properties at the mode crossing point P_1.

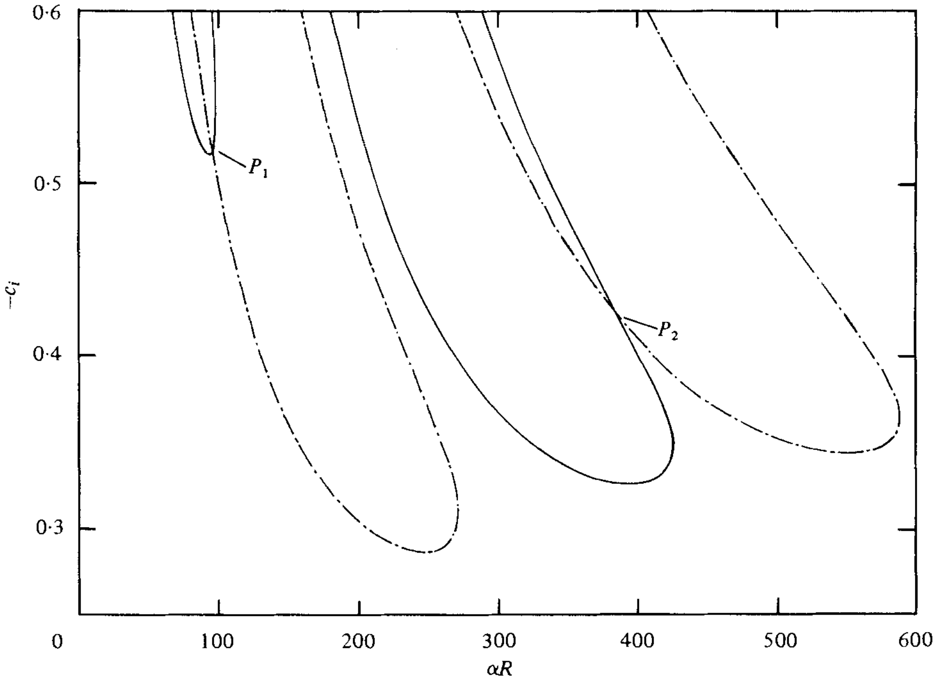


FIGURE 4. The relation between c_i and αR when $c_r = \frac{1}{2}$, $k = 5$. The solid and dashed curves represent c_1 and c_2, respectively.

experimental results are unfortunately not available which can be compared with this theoretical prediction.

The initial linear growth may lead to amplitudes so large that nonlinear effects become important. One can, at this stage, only speculate about the nature of these nonlinearities. They may be in the form of secondary instabilities or generation of large scale secondary flows. Also, because of induced changes in the mean velocity profile, a detuning of the resonance is possible.

Elongated structures have been found experimentally in other parallel flows. Such structures have been observed in the vicinity of turbulent spots in a laminar boundary layer (Elder 1960; Cantwell, Coles & Dimotakis 1978). The measurements of Wygnanski, Haritonidis & Kaplan (1979) showed that thin vertical shear layers are present at the wing tips of a turbulent spot. The observations of Komoda (1967) indicated that

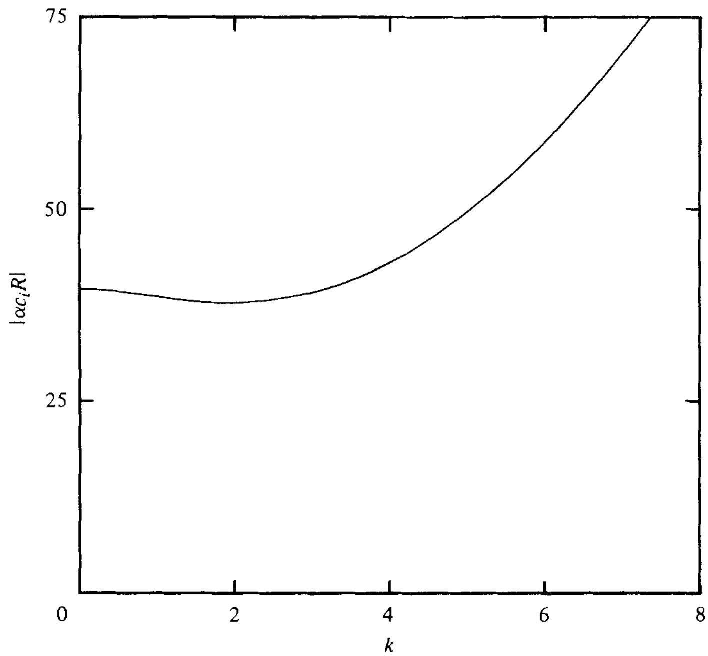


FIGURE 5. Exponential damping ratios at the resonance point P_1 as a function of k.

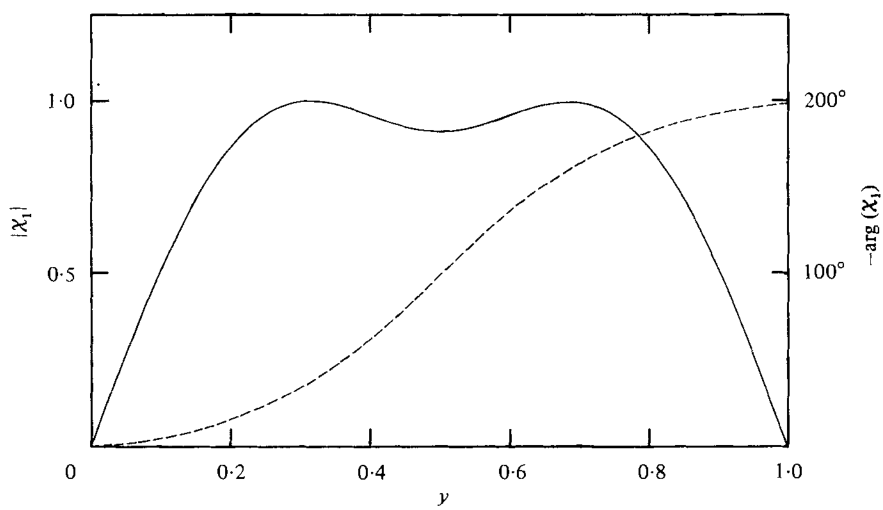


FIGURE 6. The eigenfunction χ_1 at the mode crossing point P_1, $k = 2$. The solid and broken curves represent the amplitude and phase distribution, respectively.

such shear layers precede breakdown to turbulence. The similarity between the theoretical predictions for plane Couette flow and the observed phenomena in boundary layers is striking. It is therefore of interest to assess whether a similar resonance mechanism, such as the one discussed in this paper, is also present in a laminar boundary layer. For this type of flow, numerical calculations indicate that wavelike solutions to the homogeneous operator in (7) exist. Resonant driving by Tollmien–Schlichting waves is therefore plausible. Also, the transient part of the solution to the initial value problem for the vertical velocity component (Gustavsson 1979) causes a transient response in the vertical vorticity or, equivalently, in the horizontal velocities. Preliminary calculations show that in the limiting case of α equals zero, the streamwise velocity component grows linearly for small times. Eventually, viscous dissipation will become dominant and the disturbance decays. This indicates that a similar temporal behaviour, as discussed above for plane Couette flow, is also possible in a boundary layer.

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